Pre-class Warm-up!!!

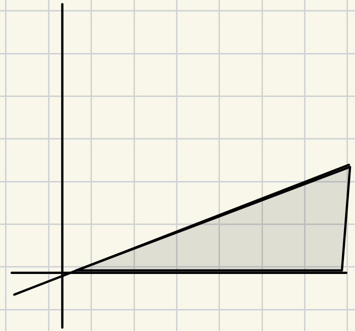
Which picture shows the region of integration of the double integral

$$\int_0^1 \int_{2x}^2 e^{\sin(xy)}\, dy\, dx$$

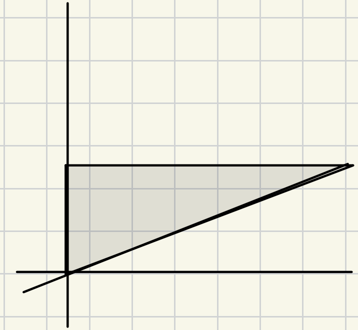
First integral: y goes between $y = 2x$ and $y = 2$

If you are asked to sketch a region of integration put in enough detail so that things are clear.

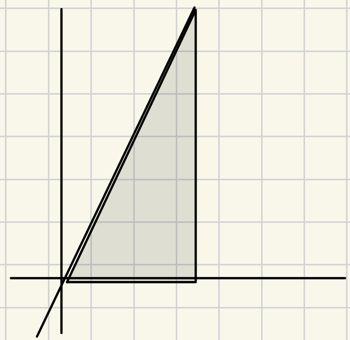
a.



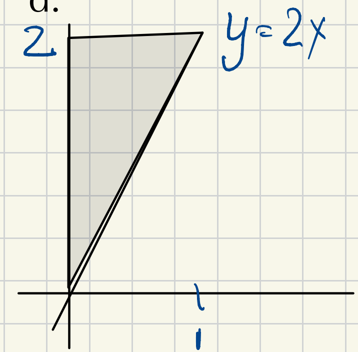
b.



c.



d.



e. None of the above.

Section 5.5 The Triple Integral

We learn

- The triple integral over a rectangular box is defined by Riemann sums.
- The integral satisfies similar formal properties to the double integral.
- We can extend the triple integral to elementary regions in R^3.
- We get practice doing triple integrals over more complicated regions.

Typical questions:

a. Like 5.5 questions 3, 4: Perform the integration over the box indicated

$$\iiint_B xy \, dx\, dy\, dz \quad \text{where } B = [0,1]\times[-1,2]\times[1,2]$$

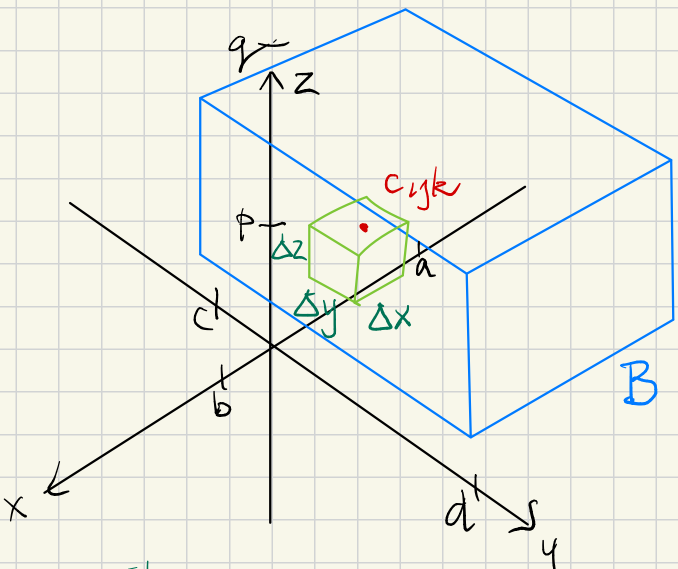
or $\iiint_B xy \, dV$. or $\int_1^2 \int_{-1}^2 \int_0^1 xy \, dx\, dy\, dz$

b. Like 5.5 questions 11, 12: Find the volume of the region bounded by the surfaces

z = 2y, z = 6-y, y = x^2, y = 2-x^2

Riemann sums

These are similar to the 2-D case. We have a rectangular box
$B = [a,b] \times [c,d] \times [p.q]$ and a function $f(x,y,z)$



We divide each coordinate B into small intervals of length Δx, Δy, Δz

In each small box we take a point c_{ijk}. A Riemann sum has the form

$$\sum_{ijk} f(c_{ijk}) \Delta x \Delta y \Delta z$$

If these sums approach a common value d as $\Delta x, \Delta y, \Delta z \to 0$, letting c_{ijk} take all possible choices we say f is integrable, integral d.

Notation: $\iiint_B f \, dV = \iiint_B f \, dx\,dy\,dz$

$$= \int_p^q \int_c^d \int_a^b f \, dx\,dy\,dz$$

Properties:

- Integrals over a box can be evaluated as iterated integrals.
- Fubini's theorem
- Things like the upper and lower bounds, the mean value theorem

$$\iiint \underbrace{a}_{\text{numbers}} \overbrace{f}^{\text{functions}} + \underbrace{b}_{\text{numbers}} \overbrace{g}^{\text{functions}} \, dV = a\iiint f\, dV + b\iiint g\, dV$$

Typical questions:

a. Like 5.5 questions 3, 4: Perform the integration over the box indicated

$$\iiint_B xy \; dx\, dy\, dz \quad \text{where } B = [0,1]\times[-1,2]\times[1,2]$$

or $\iiint_B xy\, dV$.

Solution: We calculate

$$\int_1^2 \int_{-1}^2 \int_0^1 xy \, dx\, dy\, dz$$

$$= \int_1^2 \int_{-1}^2 \left[\frac{x^2 y}{2}\right]_0^1 dy\,dz = \int_1^2 \int_{-1}^2 \frac{y}{2}\, dy\, dz$$

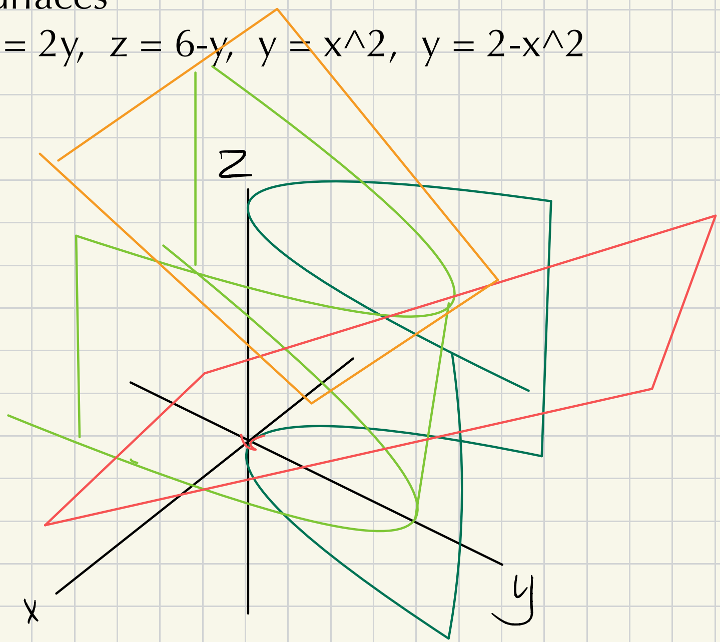
$$= \int_1^2 \left[\frac{y^2}{4}\right]_{-1}^2 dz = \int_1^2 \left(1 - \frac{1}{4}\right) dz$$

$$= \left[\frac{3z}{4}\right]_1^2 = \frac{6}{4} - \frac{3}{4} = \frac{3}{4}$$

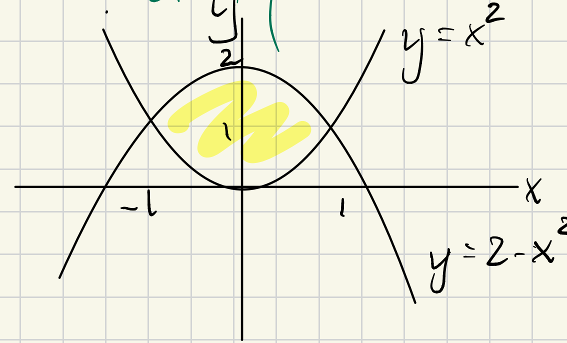
Elementary regions

b. Like 5.5 questions 11, 12: Find the volume of the region bounded by the surfaces

$z = 2y, \;\; z = 6-y, \;\; y = x^2, \;\; y = 2-x^2$



This is a strip with cross-section



Integrating first w.r.t. x or y is bad

Use z first. Volume =

$$\int_{-1}^{1}\int_{x^2}^{2-x^2}\int_{2y}^{6-y} dz\,dy\,dx$$

Pre-class Warm-up!!!

Match integral (a) to the correct region of integration

y goes from $y = -\sqrt{9-x}$ to $y = \sqrt{9-x}$

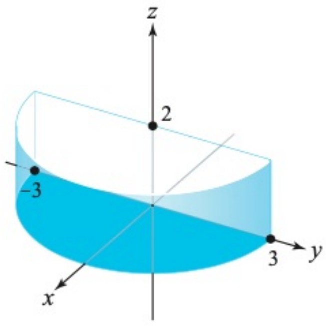
Answer: (i) (ii) (iii) (iv)

From now on Mathematica labs will be due on Wednesday 11:59pm instead of Tuesday.

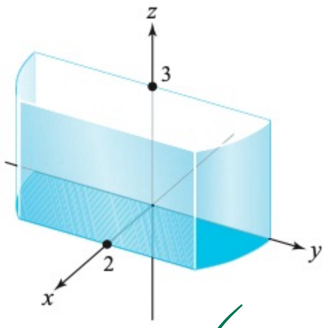
1. In parts (a) through (d) below, each iterated integral is an integral over a region D. Match the integral with the correct region of integration.

(a) $\displaystyle\int_0^2 \int_0^3 \int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\,dz\,dx$ (c) $\displaystyle\int_0^1 \int_0^x \int_0^y dz\,dy\,dx$

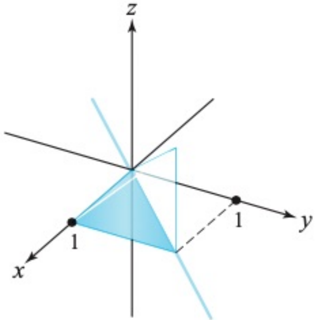
(b) $\displaystyle\int_0^2 \int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz$ (d) $\displaystyle\int_0^1 \int_0^y \int_0^x dz\,dx\,dy$



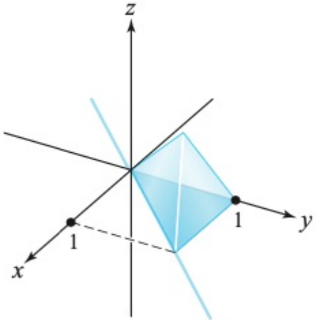
(i)



(ii) ✓



(iii)



(iv)

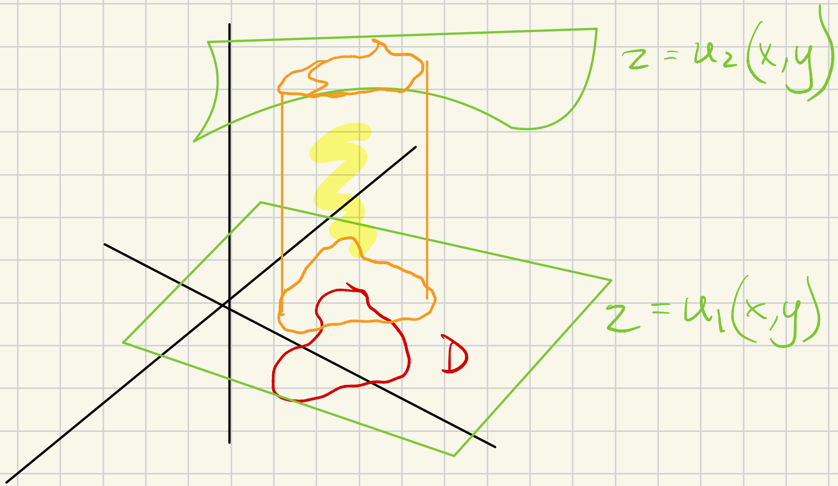
2. Evaluate the following triple integral:

$$\iiint_W \sin x\,dx\,dy\,dz,$$

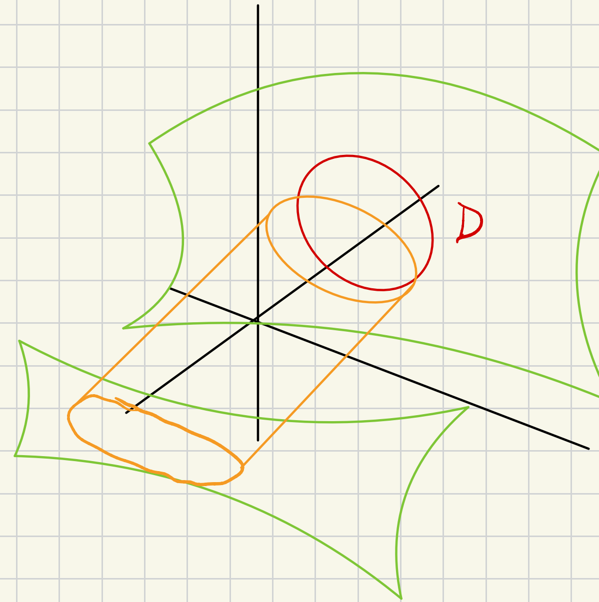
where W is the solid given by $0 \le x \le \pi$, $0 \le y \le 1$, and $0 \le z \le x$.

Elementary regions are regions of $\mathbb{R}^3$ between two graphs of functions like

$z = u_1(x,y)$ and $z = u_2(x,y)$



The orange region is z-simple.



This region is x-simple.

Elementary regions.

Which variable would you use to integrate first in (i), (ii)?

In (ii), integrating first w.r.t x is not so convenient. The integral would break into 3 pieces.

In question 1, which of (i), (ii), (iii), (iv) are elementary? With respect to which of x, y, and z?

Did you find any region and variable so that it is not elementary?

Yes

No

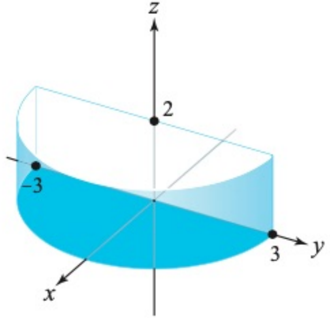
1. In parts (a) through (d) below, each iterated integral is an integral over a region D. Match the integral with the correct region of integration.

(a) $\displaystyle\int_0^2\int_0^3\int_{-\sqrt{9-x}}^{\sqrt{9-x}} dy\,dz\,dx$

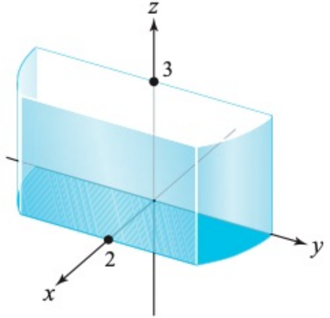
(b) $\displaystyle\int_0^2\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx\,dz$

(c) $\displaystyle\int_0^1\int_0^x\int_0^y dz\,dy\,dx$

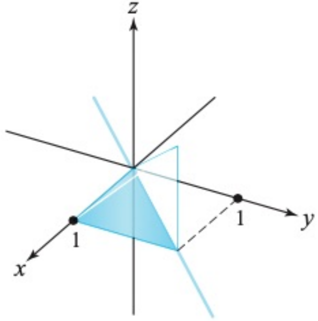
(d) $\displaystyle\int_0^1\int_0^y\int_0^x dz\,dx\,dy$



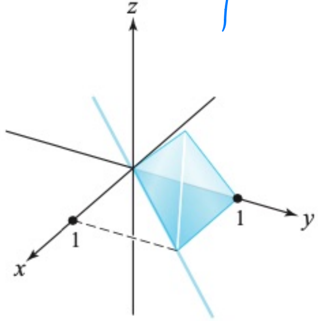
(i)



(ii)



(iii)



(iv)

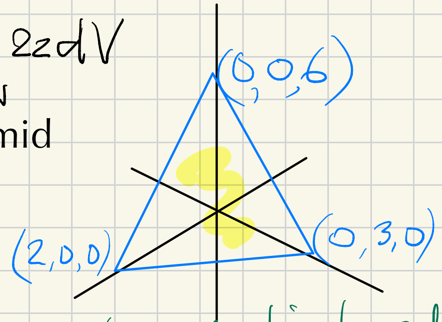
2. Evaluate the following triple integral:

$$\iiint_W \sin x\,dx\,dy\,dz,$$

where W is the solid given by $0 \le x \le \pi$, $0 \le y \le 1$, and $0 \le z \le x$.

Example. Find $\iiint_W 2z\,dV$

where W is the pyramid



We write this as an iterated integral

$$\int_0^2 \int_0^{3-\frac{3x}{2}} \int_0^{6-3x-2y} 2z\,dz\,dy\,dx$$

The blue plane is $z = Ax + By + D$.
where A = slope of the plane in direction x $= \frac{-6}{2} = -3$

B = slope of the plane direction $y = \frac{-6}{3} = -2$

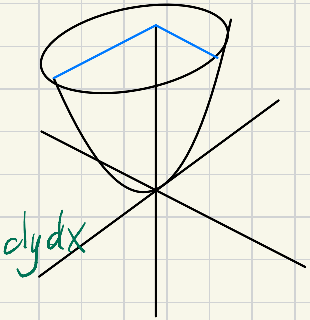
$z = -3x - 2y + D = -3x - 2y + 6$

The plane is z = 6-3x-2y
The bottom edge is y = 3 - 3x/2

Answ. 9

Example. Find $\iiint_W 3xy\,dV$

where W is the region bounded by the plane z = 2, and the surface z = x^2 + 2y^2 in the region x > 0, y > 0



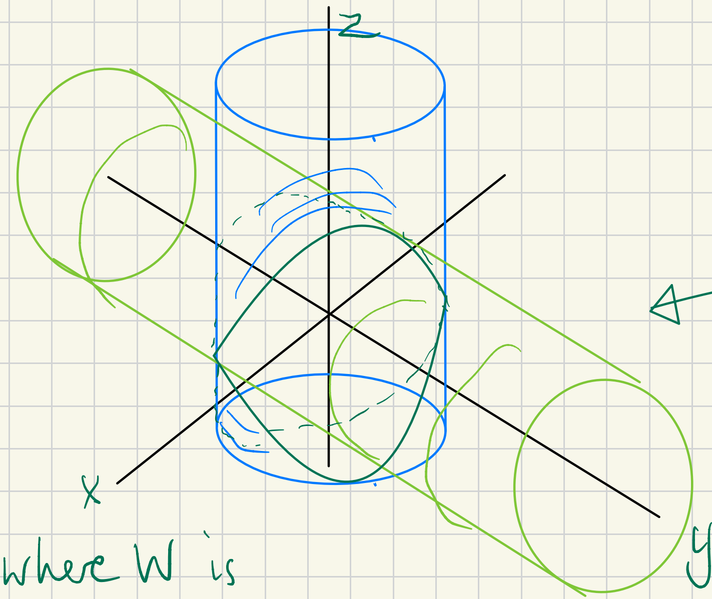
It is

$$\int_0^{\sqrt{2}} \int_0^{\sqrt{1-\frac{x}{2}}} \int_{x^2+2y^2}^{2} 3xy\,dz\,dy\,dx$$

Answ: $\frac{1}{2}$

Like question 14.
Which variable would you choose to integrate with first?

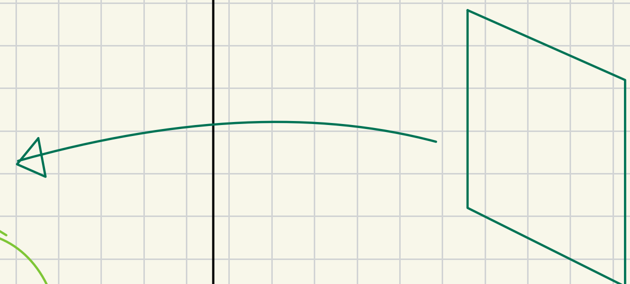
We do $\iiint_W f \, dV$



where W is the intersection of two cylinders of radius a.

The cross-section perpendicular to the x-axis is a square

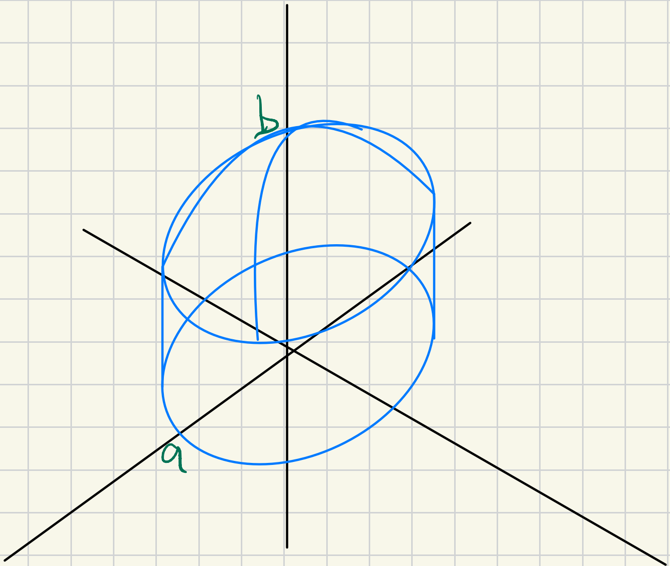
Do y, z first, then x.



We get

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} f \, dy\,dz\,dx$$

Like question 27. Which variable would you use to integrate first?



A piece of a sphere on top of a cylinder

$x^2+y^2 \le a^2$, $z \ge 0$, $x^2+y^2+z^2 \le b$

z first is most convenient.

$$\int_{-a}^{a}\int_{-\sqrt{a-x^2}}^{\sqrt{a-x^2}}\int_{0}^{\sqrt{b-x^2-y^2}} dz\; dy\; dx$$